THE

MATHEMATICAL GAZETTE.

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MATHEMATICAL ASSOCIATION.

REPORT OF THE COUNCIL FOR 1908.

THE Council deeply regret that on the 13th of August the Association lost one of its most eminent members by the sudden death of the REV. CHARLES TAYLOR, D.D., Master of St. John's College, Cambridge. Dr. Taylor had for many years been an honorary member and had served the office of President of the Association in the old days when it was known as the A.I.G.T. It is with much regret too that the Council has heard that on the 19th of December, one of the Vice-Presidents of the Association, MR. GEORGE HEPPEL, passed away at the ripe age of 78. Mr. Heppel was one of those, of whom but few now remain, who in 1871 launched the A.I.G.T. on its career, and, so long as health permitted, he actively promoted its prosperity and welfare.

The Council are pleased to be able to report that since the last annual meeting 97 new members have been elected, of whom 41 are ladies. The Association consists now of 508 members, of whom 8 are honorary members, 27 are life members by composition, 40 are life members under a rule which has now been rescinded, and 433 are ordinary members.

Early in the past year a small body of members and others, residing in the neighbourhood of Bangor, applied to the Council for formal recognition as a Branch of the Association. Their rules having been submitted to the Council and approved by them, the Council resolved to recognise this body as the North Wales Branch. The Honorary Treasurer and Secretary of the Branch is Mr. T. G. Creak, of Bryn Eryri, Llanberis. The Council welcome this new departure, and hope that similar branches of the Association will be established in other places.

It was one of the objects of this Association, when it was re-constituted, that periodical meetings of the members should be organised whenever and wherever a few could arrange conveniently to come together.

A letter was received last March from the Secretary of the Public Schools Science Masters' Association, in which it was proposed that the annual meetings of the two associations should be held conjointly in 1909. The proposal fell through for this year, as it was found impossible to make the necessary arrangements; but the Council agreed to recommend to this Association that it should hold its annual meeting in 1910 in conjunction with the Public Schools Science Masters' Association. The ordinary business might be carried through and the Presidential address might be delivered at a private meeting in the morning, and a joint meeting for the discussion of some question of common interest to both associations might be held in the afternoon. A special joint committee of the two associations would be formed to make the necessary arrangements.

The reference to the committee would be (i) to prepare a report: (ii) to consider the best means of promoting co-operation between the teachers of Mathematics and of Natural Science and of co-ordinating their methods. These subjects are to be considered generally but with special reference to drawing up a programme for the joint meeting in 1910: (iii) to collect data as to the age and stage at which boys now begin the study of (*a*) Practical Geometry; (*b*) Practical Measurements; (*c*) Practical Physics and Chemistry, and if necessary to propose any changes which may seem desirable: (iv) to attempt to co-ordinate the elementary work of the two classes of teachers by (*a*) drawing up a syllabus which shall satisfy both Science and Mathematical Masters in making Practical Mathematics a preliminary to Practical Science; (*b*) considering how far it is possible to introduce common definitions and to make the examples and problems mutually helpful: (v) in higher work to consider; (*a*) the stage at which and the method by which the use of logarithm tables and of the slide rule may be best introduced; (*b*) the teaching of Mechanics both Theoretical and Practical.

The Council propose that this Association be represented on that committee by Messrs. J. V. Coates (Alleyn's School), C. Godfrey (Royal Naval College, Osborne), C. S. Jackson (Royal Military Academy), D. B. Mair (Civil Service Commission), A. W. Siddons (Harrow), and F. J. W. Whipple (Merchant Taylors'). The Council of the Public Schools Science Masters' Association have nominated Messrs. Berridge (Malvern), Borchardt (Cheltenham), Cumming (Rugby), Garbutt (Winchester), Thwaites (Leicester) and Vassal (Harrow) as their representatives. Two

additional members Messrs. C. C. Lynam and J. S. Norman have been nominated by the Association of Headmasters of Preparatory Schools.

On the 28th of November a meeting was held by the Mathematical Association, in conjunction with the Federated Association of London Non-Primary Teachers, at the Polytechnic, Regent Street, at which Professor John Perry, F.R.S., gave an address on "The correlation of the Teaching of Mathematics and Science," and a discussion followed. A verbatim report of the proceedings, including Professor Perry's address, the speeches which followed and Professor Perry's reply will be published shortly in a special number of the *Mathematical Gazette*. The meeting was attended by more than 150 persons, and was the largest in which this Association had taken part. The Council desire to express the Association's most cordial thanks to Professor Perry for the address which he gave on that occasion.

The stock of the "Report on the Teaching of Elementary Mathematics" being nearly exhausted, the Report has been reprinted, together with the other reports which have been published by this Association during the years 1902 to 1908. This re-issue, in one cover, can be obtained at sixpence per copy, from the publishers, Messrs. George Bell & Sons.

The compounding fee for life membership was fixed at five guineas at a time when the annual subscription was seven shillings and sixpence. The annual subscription now being ten shillings, the Council recommend that the compounding fee be raised to seven guineas from the date of this meeting, and that the rules be altered accordingly.

The President, Professor G. H. Bryan, has now completed two years of office, and the Council desire to record their recognition of the debt which the Association owes to him for his activity in promoting its objects. The two years have been years of progress, largely due to the work which the President has himself initiated, and in which the Council has had the benefit of his active co-operation.

In succession to Professor Bryan, the Council nominate Professor H. H. Turner, F.R.S., Savilian Professor of Astronomy, Oxford, as President of this Association for the years 1909 and 1910.

The Council propose also that Professor G. H. Bryan, F.R.S., be elected a Vice-President, and that Professor John Perry, F.R.S., be elected an honorary member of the Association.

Mr. C. Hawkins and Mr. C. A. Rumsey now retire from the Council in accordance with the rules, and nominations of two members to occupy the vacant seats will be called for at this meeting.

The Council desire again to express their cordial appreciation

of the services of Mr. Greenstreet as Editor of the *Mathematical Gazette*. Three extra numbers of the *Gazette* have been issued during the last twelve months, and Mr. Greenstreet's work has consequently been heavier than usual.

The Council again desire also to express their thanks to Mr. C. S. Jackson for the admirable arrangements which he made for the fourth annual dinner last January: to the Council of King's College for their kindness in granting to the Association all the accommodation which it has needed for its meetings: and to the Council of the Polytechnic for its hospitality on the 28th of November.

THE ADDRESS OF THE RETIRING PRESIDENT.

THE Report of the Council bears abundant testimony to the ability and energy which that body has displayed in increasing the usefulness of the Association, in particular in bringing it into closer touch with outside interests and other bodies. If this work is continued in the future as it has been of late, there is every prospect that in the course of time our Association will rise to the status and influence of the American Mathematical Society or the Deutsche Mathematiker Vereinigung, and this result will well repay the hours of work and discussion that are undertaken by the body of councillors meeting periodically at King's College.

In this address it is proposed to deal with some of the more serious consequences of England's neglect of Mathematics with especial application to the problem of teaching.

Before doing so it appears desirable to refer to several current topics, and I wish at the outset to say something about our North Wales Local Branch, in the hopes that others will "please copy."

The success of this branch is due to our local secretary, Mr. T. G. Creak, an old Cambridge wrangler and retired schoolmaster. We have over a dozen members and associates, including teachers in secondary boys' and girls' schools at Beaumaris, Bethesda, Colwyn Bay and elsewhere, and an elementary teacher from our immediate neighbourhood who has greatly contributed to the value and interest of our discussions by representing a phase of teaching with which it is often extremely difficult to get into proper touch. We have held meetings up to now three times in the year, and on such occasions these hard-worked schoolmasters give up their Saturday half-holiday and come over by train or boat or on foot to the meeting place, generally Bangor. As a rule, one member is asked to open a discussion on some subject, such as teaching of algebra or the extent to which text-books

should be used in teaching. Then we sit round a table and talk. Every meeting impresses me more and more with the originality, ability and general interest shown in their work by those teachers who take part in these discussions. The opening papers have been excellent. In the discussions that have followed the noticeable feature has been that every one has much to say, and everything said has been well worth saying. For example, the conversation turns on the subject of how best to teach a beginner that the product of two negative quantities is positive. Then one of our members puts forward a method he has thought of identical in principle with that used in interpreting complex quantities, and the idea is at once taken up by the other members. One point which these discussions have brought into prominence is the fact that our schools have to prepare for examinations conducted by at least four independent bodies, each under separate control, and each appointing outside examiners who are as a rule not in touch with our system of education. The teachers themselves are willing enough to adopt more practical and intelligent methods of teaching; indeed, they have gone very far in this direction, but the examiners will insist on the old-fangled, useless drudgery, and the efficiency of the classes suffers in consequence. One master was taken to task by his inspector for bothering his boys with simplifying complicated and meaningless algebraic expressions, whereupon he took down a book of recent examination papers and showed the inspector a question of the kind in every one of them.

I give this account mainly as an example of what a local branch can do. It is remarkable what a lot of ideas have been brought up in our informal discussions. At the end of every meeting we have held over abundant material for the next meeting. Surely local branches in other parts of Great Britain would serve an equally useful purpose. There must be people elsewhere who could devote the amount of time necessary for the duties of local secretary, even if they are not quite so free in this respect as our present secretary.

The recent meeting at the Regent Street Polytechnic, at which we had the pleasure of listening to Prof. Perry, affords another illustration of the activity of our Association. On reading over the reports of the proceedings the one point about which I am in doubt, despite my attempt to save this particular situation, is how far the members of our Association fully realise the serious danger which is imminent at the present time, namely, the extinction of the English mathematical specialist. We were told that mathematicians must take off their coats and work. I would say that English mathematicians must take off their coats and fight for their lives. At present no one in England has a good word to say for the mathematical specialist. In other

countries he is exercising an ever growing influence on the destinies of the nation. We were told at that meeting that to put a mathematical specialist on to teach elementary pupils is like putting a steam hammer on to crack a nut. But the nation which tries to do without the mathematical specialist is like the foundry where they try to drive rivets into boiler plates with a pair of nutcrackers. We still have plenty of steam hammers and rivetting machines, which have to content themselves with cracking nuts, as the supply of steam is not sufficient for anything more. Meanwhile, if things go on as they are going, we shall have to buy all our boilers from the German makers. Our present Chancellor of the Exchequer has learnt that if you go over a works in Germany they will probably put a university professor on to show you round.

I have gone very carefully into the question as to how far a mathematician who has had no practical experience can be of use in solving the problems presented by the applications of science to modern industry. The opinion I have formed is that there is plenty of work to keep mathematicians continually employed in their studies without their having a minute to spare to use their hands except for the purpose of writing. Unfortunately, however, *this work doesn't pay*. I do not mean that the mathematician cannot make a fortune by doing the work. What I mean is that he will lose the small amount of compensation that he would obtain by giving the same number of hours to elementary teaching, and he will wear himself out far quicker than he would do if he were to forget his mathematics and to level himself down to the standard of a common or garden all round teacher. A man with a sufficiently large unearned income saved by his parents could greatly advance matters, but it is the will of our nation to place heavy obstacles in the way of the man who attempts to live on such an unearned income, even if he gives more than its equivalent back in unpaid work.

The story has been repeated *ad nauseam* of the mathematician who made a discovery, and who stated that its great charm was that it could never be of any use to anyone. There are two sides to that story, and the commonly accepted version is the wrong one. The mathematician probably knew that he would get the kicks in any case. What pleased him was the hope that the man who gave him them would not pocket the halfpence. It is quite possible, moreover, that if the mathematician lived to the present day he would find his discovery put to practical use.

Any mathematician who will try and get into contact with the English practical man will discover widespread ignorance of the most elementary ideas. Not very long ago I found in the columns of the *English Mechanic* the old exploded fallacy about the oar being a lever of the second class, and numerical results

given which were based on this fallacy. It never occurred to these good people that had their calculations been correct, the simplest way of propelling a boat would be to do away with oars altogether and to sit in the seat and tug away at the row-locks. A man had a gas engine and a well, and he wanted to pump water out of the well with his gas engine. He consulted a "practical" engineer, and was told this was impossible. So he himself had to get the belting and pulleys necessary to gear the engine down. He invited the engineer to see the machine actually at work. I believe the engineer still retained the same scepticism that the old Pisa professors had after they had seen Galileo's stones dropped from the Leaning Tower.

English firms who have engines ordered abroad for special gradients send ordinary engines instead, simply for want of a little elementary mathematical knowledge.

I believe that the hard things which are said about mathematicians in England are due to the fact that people in general cannot distinguish—it is impossible that they should distinguish—between a wrangler and a mathematician. By a wrangler I mean not only a Cambridge wrangler, but any man who undergoes a rigid course of mental training to enable him to pass a difficult examination in mathematics. One peculiarity of this examination is that the syllabus goes a long way past the dividing point at which the mathematician and the physicist branch off in different directions. Another peculiarity is that it covers a number of subjects each of which, to be properly mastered, would take the whole number of hours that the candidate usually can devote to the entire syllabus. A third peculiarity is that greater importance is attached to the candidates' powers of devising artificial dodges for the solution of tricky problems than to his knowledge of first principles and their direct applications. The examinations set for the Indian Civil Service illustrate these peculiarities in a considerable degree. I believe that if the candidates, instead of having to face these advanced papers, were required to show a thorough knowledge of *one* such subject as Theory of Groups, Higher Analysis or Theory of Statistics, and Probability, they would be much better fitted for the Indian Civil Service. In the last examination only six out of the thirty-five top candidates took Advanced Mathematics, while twenty-four took Greek History.

Personally, I think that properties of circles, triangles, and trilinear co-ordinates are carried too far, both in our *Gazette* and elsewhere. They do not produce the kind of student who can solve at a glance a very simple problem on the catenary or cycloid, such as every practical man should be able to solve, without using pages of algebraic formulae.

The dynamics of the typical wranglership examination can

best be described as "Dogmatics." I have myself to teach students this subject for examinations of the wranglership type. The typical wrangler believes or tries to believe that all bodies are perfectly smooth, and he has to prove, or attempt to prove, results quite different from anything which he will ever see in a physical laboratory or elsewhere.

Thus is produced a man who is called a specialist and condemned as incompetent to teach. He is a wrangler, but not a specialist.

On the other hand, a man who has taken a general practical course in mathematics and science becomes none the less a good all-round man, none the less competent teacher if his ideas have been broadened and extended by reading such a book as Hilton's Groups, or Carslaw's or Bromwich's Series, or one of the other modern treatises published at Cambridge or elsewhere. I maintain that the successful teacher requires his ideas to be enlarged in some direction beyond the syllabus of the typical B.Sc. examination, if he is not to run in blinkers all his life. If this be done, we shall train up a body of teachers each of whom is a specialist in his own little way, and some of them will in course of time do their quotum of research, but these teachers will in no case have ruined their careers by making themselves unpractical or unsympathetic with the work they have to do to earn their daily bread. They will in fact be much more like the teachers who are turned out in every civilised country except Great Britain, and who certainly succeed in producing intelligent citizens.

The specialist teaching elementary classes has been compared to the racehorse drawing a cart. If this analogy is correct, it is nothing less than sinful to bribe the horse, by offers of fodder in the form of scholarships, to train as a racehorse, if the supply of fodder is to be cut off as soon as the race has been run. The alternative I propose is to give the horse a bit of a run over some rough or newly explored piece of ground. When he has been put into the shafts, I maintain that he will be much better able than before to pull his cart over any unexpected obstacles.

It is high time that we should insist on our mathematical teachers possessing at least the knowledge of their subject necessary to read one of the advanced books referred to. A student of mine who had been teaching mathematics for some time in a school told me he was going over to Germany, and asked my advice. I advised him to follow a course of university lectures there, but he said he would not be able to understand them. Such a man is not fit to teach. Again, I constantly hear of students who have neglected their mathematics in order to take honours in Chemistry, Latin, or Botany, and who then try to teach mathematics in the schools. Then the schools wonder at their large number of failures.

The number of members of the Council and Committees of our Association who have taken Part II. of the Mathematical Tripos, bears abundant testimony to the valuable work which specialists are able and willing to do when they turn their thoughts to questions connected with elementary teaching. It would be a strong argument in defence of the specialist, if statistics were drawn up showing the proportions of Part II. men serving on these committees.*

I have dwelt on this subject, as I believe that the extermination of the specialist is one of the greatest calamities that can befall our country. Even an invasion by Germany might improve matters in this one particular respect, if in nothing else.

Coming next to the training of the ordinary citizen who goes to school—whether to the elementary or to a secondary school—and who does not intend to go to the university, nor to take up physics or engineering, what mathematics ought to be taught to him? I think we are mostly now agreed that he ought to learn to calculate and measure properly. It must not be forgotten, however, that constructive geometry can be made as dull and uninteresting as anything else. If you put a boy down to verify a lot of geometrical propositions by drawing figures with a hard pencil, where he can see by his own reasoning that the results must be true, you will be doing that boy more harm than good.

Another important feature of a boy's education is that he should be able to properly estimate the chances of an uncertain event. Everything human is uncertain. The boy who decides to embark on a certain career simply because some successful man in that profession happens to be earning £1000 a year is a gambler, pure and simple. He only ceases to be a gambler when he takes the element of chance into account, and estimates his probability of overcoming the odds that are against him by his individual exertions. I am glad to see that a chapter on probability has been introduced into a recent arithmetic book. The subject can be taught experimentally as well as theoretically. It exemplifies cases in which fractions are more useful than decimals, and it should therefore be welcomed by some of our members who have strong views on this point.

I am afraid that many of the students who are now training for elementary teachers will not only never join our local branches, but they will fail to teach our working classes how to measure and calculate properly. Till this year I have had to teach such students. They sometimes had to leave their class to pass an examination in music, and we were told that a certain Mus. Doc. had described their performance as the best he had

* Since writing this I have noticed that three out of the six delegates appointed by our Association to confer with the delegates of the Public Schools Science Masters are men who *to my personal knowledge* have distinguished themselves in Part II.

ever heard. But as for bringing the least bit of common sense to bear on a practical question, that was in many cases quite out of the question. There were notable exceptions, but their general idea was that the whole object of the mathematics classes was to enable them to answer stereotyped questions in a stereotyped examination paper.

A cook once asked us the meaning of "dactyl" and "spondee," and said that her little sister at the Board School had been asked to write a sentence containing one of those words.

The want of practical elementary experience in calculating and measuring is everywhere apparent. The tailor who cuts one's coat according to his cloth fails to make that coat fit simply because he has never performed laboratory experiments on the stretching of cloth in different directions when weights are suspended from it. He ought also to study geometrical models of the various surfaces that can be formed by the deformation of textile fabrics. I believe that in Germany such things are taught. But England is not Germany.

In conclusion, the mathematician of the future, if he is to save himself from destruction, must be a fighting man. He must fight against the mass of superstition and prejudice that has grown up against him. He will have no time to tinker with syllabuses or to squabble about petty matters of detail with his brother mathematicians. He must convince the British public that he is a power in the country, that he is anxious to use his power for the benefit of the country, and that the country cannot get on without him.

["*The Parting of the Ways.*"—The following remarks were thought of after the address was communicated; they seem however to deal with an important point which has not been previously touched.

The task which the joint Committee of our Association and the Science Masters have undertaken will necessarily involve, at some stage or other, the classification of mathematical and physical subjects into three more or less defined groups; the first containing such subjects as ought to be taught to all would-be students whether of mathematics or of science, the second including subjects required for the study of mathematics only, and the third those required exclusively for the study of experimental sciences. This classification however involves a system of watertight compartments, and is bound to be a failure if rigidly insisted on. There are very few branches of mathematics which have not *some* application, and that usually a very important one, to science; very few branches of physics, a knowledge of which is not of great use in illustration of some mathematical theory. One teacher will want one thing put in the first group, another will want another, and in the end the

curriculum instead of being simplified will be made heavier and more indigestible than before. The only remedy is to give greater scope for individuality on the part of teachers, so that one teacher may exploit a part of the subject which he considers important, while another is laying stress on another part. Instead of a hundred schools each taking its pupils over the same ground, we want them, as soon as the necessary minimum of common knowledge has been covered, to branch off on as different lines as possible, so that if a group of their pupils be selected at random after they have left school, they will be found in the aggregate to possess a wide knowledge of mathematics or physics, and not all to possess a common knowledge of an infinitesimal fraction of these subjects. No education can be efficient which does not turn out all sorts and conditions of men.]

THE MATHEMATICAL ASSOCIATION.

THE MATHEMATICAL TRAINING OF TECHNICAL STUDENTS.

[An address given by Dr. T. H. Boyey, F.R.S., to the Mathematical Association, Jan. 12th, 1909, at King's College, London.]

When your Council did me the honour to ask me to give an address at the General Meeting of the Mathematical Association of London, I felt a considerable amount of hesitation, as I know very little of your work or of the ground covered by your deliberations; but the suggestion having been made that I should take up the subject of the Mathematical Training of Technical Students, I felt that my personal experience and the necessity I had been under of keeping myself informed as to the ideas of other engineers on the subject, had forced me to form some opinions which I may now submit to you, with the regret only that the claims upon my time incident to taking up new work have not permitted me to do more than gather together results of my own experience and that of others in a rather hasty manner.

The first conclusion which I was led to by the exigencies of practical work, when I commenced my professional career under the Mersey Dock Board, was that the mathematical knowledge with which I had left Cambridge was, in some respects, not quite sufficient to meet the demands of structural design.

I soon found that I had to deal with many indefinite data and that exact calculations had to be supplemented by what have been called, with a pleasing concession to actualities, "factors of safety." I also found that the refinement of accuracy if I may so call it, encouraged by a training in pure mathematics,

was of less importance than the broad accuracy dictated by the actual factors of the problem in hand, and would only take up valuable time which would be better spent in checking the large results.

However, notwithstanding these experiences, I feel glad now to think that, in undertaking, with youthful temerity, the education of others in M'Gill University, I did realize from the first that mathematical training is, and must be, the foundation stone of the training of a good engineer—who can never rise in his profession, unless either by nature or training he has the power to grasp a subject mathematically.

Without further preamble, then, let me endeavour to set before you some considerations about the aim we should place before us, and the manner in which, in the present state of our knowledge and possibilities, we may hope to approximate to it.

Let us ask ourselves two questions:

"What effects are we trying to produce by a training in Mathematics?" "And what means are at our disposal to carry out our aims?"

I. The first aim I shall mention is that we want to teach, as it were, the mathematical language. I heard of a man the other day who, wishing to learn a difficult Eastern language, was found learning another, and in reply to an exclamation of surprise from the friend who had discovered his occupation, said calmly, "Oh yes, I shall learn it in a shorter time if I learn this first."

Whether or not we agree that the scholar knew what he was about, there can be no manner of doubt that in learning to make the measurements and calculations necessary for advance in almost every science, mathematical grasp and mathematical expression provide us with a real short cut without which we could not expect to cope with the difficulties of the task.

Secondly, we are trying to give what is conceded to be a most valuable mental discipline. Mathematics requires concentration of attention, power of reasoning, patience, and determination; and although the student often reduces the valuable mental effect to a minimum by doing just as much as, and no more than, is required to pass some examination, it cannot fail to be of some noteworthy importance among the many weapons with which Education is trying to advance the bringing of order out of chaos—the infallible sign of a real civilization.

Thirdly, we want to stimulate the mind of the student and gain his real interest, to awaken his love of knowledge, *per se*, and his delight in the use of it in the conquering of circumstances, as an expert fencer takes a positive delight in making his weapon answer his will.

Fourthly, we want to give the knowledge of principles as

connecting links and interpreters of facts, in order to make short cuts for our students in the carrying out of their work.

Fifthly, we want to guide the students into the right manner of applying principles to show them what kind of allowance must be made for the various exact or indefinite data which have to be taken into account, and, so far as possible, how to estimate the degree of effect for which they must make provision.

The last is probably the only aim about which there might be question—because in some people's opinion you cannot *teach* exceptional applications of principles. I am not prepared to agree with this opinion.

It is evident that no mathematical knowledge pure and simple will teach the power of dealing with machinery or with natural forces, hence we have the recognised necessity for such sciences as Kinematics and Hydraulics, and as it is the result of experience that a man of affairs will deal best with affairs, so a man accustomed to deal with a certain class of problems will deal best with a new problem of the same class. It may even be possible to teach the probable directions in which to look for danger.

After considering what our aims are, let us next give some attention to the *means* which are available for the carrying of them out.

We may for our present purpose enumerate these as follows:

Teachers,
Text-books,
The mental powers of the student,
Equipment, and
Carefully planned courses.

First, then, let us think of our *teachers*. I shall take for granted that they possess all the knowledge of mathematics, all the knowledge of men, all the personal influence which are always so desirable, and here I shall confine myself to the question:

"Should the Mathematical Training of Engineers be necessarily in the hands of Engineers?" I think there can be no doubt that the mathematical teacher in a technical school, whose training has been confined to pure mathematics, *is* himself, and puts his pupils, at a disadvantage. Ideally speaking, a man filling such a position should be an engineering graduate, because the teacher should be thoroughly acquainted with the problems that the engineer has to face, in order that, by a good selection of these as subjects of study, he may prevent the waste of valuable time.

Practically, however, it is difficult to get engineering graduates who can speak from the standpoint of advanced mathematics—a necessary vantage ground for the teacher.

The chief reason for this difficulty is that, probably, in the first years of development of a practical science, the tendency

is to hasten over the theoretical side—an evil which I shall have occasion to allude to later.

On the subject of mathematical instructors, I may be allowed to quote from a paper by Professor Slichter, of the University of Wisconsin:—

> "When we come to consider the make-up of the instructional force we find the very greatest difference of opinion among those who are interested in the subject. Personally, I hold to the opinion, which has been so well expressed in recent editorials in *The Engineering News*, that the instructional force teaching mathematics to engineering students should be made up of men that have themselves taken a technical course. However, when we come to apply this principle under existing circumstances, we find the greatest difficulty in its practical application. In order to be competent to carry on instructional work of this kind, an engineering graduate should have had at least three years of graduate work in mathematics. Under existing conditions, the graduates of technical schools are unwilling to give this amount of time to their preparation for instructional work. In fact, it is exceedingly difficult to attract any of the graduates of the technical schools to instructional work of any kind. The opportunities for a successful career in technical work have been so great that the appeal is almost entirely to the professional ambition of the young man. On this account, it seems hopeless to secure technical graduates as instructors who have made adequate preparation in mathematics."

As a suggestion, where some practical decision must be made, I would say that when the difficulty referred to exists, that is, when it is impossible to procure an engineer having the requisite mathematical attainments, the best compromise at present is to secure an excellent mathematician and induce him to fit himself for the post by making himself in some degree familiar and sympathetic with the engineer's point of view, and with the class of problems with which his students will have to deal in after life.

II. We are next to consider the means at our command in the matter of *text-books*.

A mathematical text-book is really a symposium of the work and experience of many men. Some people go so far as to think it is of much greater importance in University work than the teacher or lecturer.

All, I think, agree that the text-books of mathematics used in Engineering Schools should be specially written for the purpose, mainly that there may be a new choice of subject

matter, and perhaps an entirely new arrangement of the course.

There has of recent years been a great improvement in text-books for engineers, and even an effort to carry out this suggestion of making them text-books of "engineering mathematics," especially in a set of text-books called "Practical Mathematics." No doubt the supply will soon meet the demand.

I think the chief difficulty at present to the author of a text-book, is whether his book is to have a character which we might call "encyclopaedic" and contain everything, or at least suggestions of everything that is known about that subject, or whether it is to be limited to what the author thinks desirable for the training of an average student.

The only regular form of limitation so far observable, is the limitation of age. Fortunately, school-books are graded; but when we come to University text-books, there is no regular system of limitation, and it may well be considered whether it is not time to bring out—by the united efforts of many scholars of course—an encyclopaedia of Mathematics, and to relieve the text-books of that mass of material which may be said often to bring an additional difficulty—as well as help—for the student, and always to tax his pocket.

In the meantime, at least, and even in a more ideal condition, I cannot agree with the writer who says that text-books are all-sufficient, and who would confine the function of the teacher to that of a kind of machine for keeping order, and spurring laggards, and seeing that the work is understood—and done.

The teacher's business is more complicated than this, and even in this special matter of text-books he is all-important, as has been said, in guiding the student to the essential and to the leaving out of the non-essential.

III. Let us consider the means at our command in the proper enlisting in the service of our work, of the powers of the student's own mind, so far as these are common to all minds.

As yet, very little attention has been given to this subject, chiefly because the pressure of work and the difficulty of arrangement are so great, except, indeed, in so far as individual professors have worked it out for themselves.

In one aspect, this is all as it should be. The photographer takes a world of trouble to sensitise his plate, but here we are dealing with a living organism, which can, and does, exert its own powers.

Still, even in the case of appetite for food, which is fairly developed in the average person, we think it worth while to stimulate desire by what we call *appetising* food—so surely it might be well, in presenting to the brain its normal food of knowledge, to serve it in such a way as to excite interest, which

might be called the appetite of the mind, and without which assimilation is difficult, if not impossible.

In fact, the motives to mental action may be said to be, every one of them, varieties of interest, if not in the work itself, then in the making of one's career, in the performance of one's duty, in the desire to please someone.

Now my experience is, that the percentage of students who take a real pleasure in their mathematical work is rather small, but I believe it could be greatly increased by some modification of the work. Some effort, for instance, not to let the pressure of speed be *always* present—an ever-looming nightmare we might call it. I would suggest that training in speed, which is far too important to be omitted, should be separated to some extent from other kinds of training, and should be carefully proportioned to the training in accuracy, with which it is very apt to interfere,—and that both should be subordinated at other times to the best manner of solving a problem.

That this would not be proceeding on wrong lines, I think should be evident when we consider that, after all, it is only giving, at well-defined times, the first place in importance to the judgment and imagination of the student—qualities without which the important qualities of accuracy and speed are both of little avail.

A spirit which it would be worth while to exorcise, if possible, is that Mathematics is an unnecessary bugbear. It will be found well to appeal to the intuition, as often as possible, to guard against the idea that mathematics is gratuitously complicated, and to cultivate the idea that it is a key to unlock the riddles of Nature. If this is successfully done it cannot fail to render the study of the necessary mathematics much more interesting.

IV. Now let us take up the question of the mechanical means at our disposal, namely, Laboratories and Class-rooms, shall I add, in order to throw in a word as to the impossibility of getting the best mental results in vitiated, over-chilly, or over-warmed air, or in the midst of noise and confusion.

However, as this may be considered self-evident, and therefore requiring no attention, (perhaps from a slight confusion between the words "self-evident" and "automatic"), I shall go back to Laboratories.

Here the essential question is, I suppose, "Would it be desirable to have a perfect laboratory if it could be secured?" It is, of course, a well-known fact that good work, and often the best work, has been done with small means, and that to some extent the absence of means rouses the dormant powers of the mind; but in the first place what might be all right for one person is of no use for many. One student may with advantage be allowed to look about for his materials and contrive his own apparatus,

but twenty doing the same thing would produce dire confusion and no results. Also what the genius will accomplish is no standard for the average man; therefore I say, have Laboratories as perfect as possible—in kind. There is no necessity, however, to have more machines than will exemplify every type of work. For this reason, training can often be given from an old machine, so long as its type is not obsolete.

As to the scale upon which machines should be made, my own opinion is that they should be of a practical size, and not mere models, because, to mention one thing, there are many cases in which size brings in different conditions which have to be taken into account.

As a matter of practical politics, I have found manufacturers very willing to present apparatus, or give their best terms to institutions, being actuated both by public spirit and by the certainty of thus obtaining a means of making their machines known.

In these last remarks, I am of course thinking mainly of the laboratories where the problems of *Applied Mathematics* would be studied.

One thing is quite evident by these remarks, I hope, namely, that a Laboratory should be endowed or supported with a sufficient sum to keep it from falling into the category of a museum.

V. We now come to the fifth means at our command for securing the various aims which we proposed to ourselves—namely, *Well-planned Courses.*

This is a subject on which there is not only too much to say this afternoon, but one on which there always will be too much to say. I shall only attempt to indicate some of the factors which should always be taken into consideration.

First and foremost, the relation of the work in which mathematics takes the position of importance to the remainder of the student's work.

This can only be satisfactorily decided after taking into account some larger problems, namely:—the relation of the average student's whole work to his circumstances, especially to his degree of preparation—to the height of the standard to which you expect him to attain—to the necessity or non-necessity for an early start in his career—to his financial abilities and, in general, to common-sense.

Having estimated the claims of all these, and found what time is at our disposal, we shall probably still give a large portion of our attention to mathematics, and the mathematical treatment of problems, on account of intrinsic importance and because the tendency to which I have before alluded, namely, to hurry over the theoretical on the supposition that we are thereby

proving ourselves practical, is so marked that there is little danger of erring in the direction of giving too much time to theory.

Next we must consider in what proportion the mathematical time should be divided between mathematics pure and applied. This again depends on several things.

If we can secure the putting of the *emphasis* on the *methods* in the study of applied mathematics, we may safely allow it a much larger place in our curriculum than would otherwise be possible.

To make some sort of decision, I would suggest taking the minimum amount of mathematics necessary for the future requirements of the student (which minimum must definitely include calculus) and teaching *that* as pure mathematics in the most satisfactory manner possible, then adding to the student's knowledge in any department, and to his knowledge of the relations of the different departments, by much drill in those problems of applied mathematics which he may be expected to meet in his profession.

This will, I think, prove a sufficient concession to the doctrine of immediate usefulness which is naturally so common to youth and to grown-up inexperience, but it is more than likely that, in order to cope with the unwillingness, the "I don't see the use" spirit, which springs from the false doctrine of utility as opposed to the true—it is more than likely, I say, that it will be necessary to spend some little time in *directly* rooting this idea out of the student's mind and getting him to relegate it to the limbo of the unwise.

Having decided, then, on the relation of mathematical work, as such, to the whole work, and also on the proportion of Pure Mathematics to Applied Mathematics, the next subject would be the proper subject-matter of the course in Mathematics.

The minimum of such a course should include, I believe—Geometry, Algebra, Trigonometry, Analytical Geometry, Calculus and some Higher Mathematics—the last at any rate in the case of advanced students.

If they have not already been included in Matriculation subjects, the course should begin with geometry and algebra, the former at first being given about double the time of the latter. As soon as the essential foundations of algebra are mastered, trigonometry should be added, perhaps, as far as the solution of triangles, then bring in analytical geometry, but omit, at first, at any rate, all discussion of tangents and other properties of conic sections and indeed of all kinds of curves.

Let the calculus be introduced at this point, for the reason that, as soon as the student understands what is meant by differentiation, he understands the shortest method of finding equations

to tangents, their slopes, etc., and thus avoids the somewhat lengthy and cumbersome methods adopted in analytical geometry. This advantage is gained, of course, by means of the use of the infinitesimal triangle. Further elaborations of analytical geometry may be introduced if time permits, and if such a course is justified by the natural powers of the student.

In speaking of the calculus, I thoroughly believe in teaching the differential and integral calculus together, and emphasising the relation between them.

The Higher Mathematics required for the study of Elasticity, of the whirling of shafts, of certain Electrical problems, of Thermodynamics, Vibration, etc., which may be undertaken by advanced students, are probably the advanced parts of calculus, Differential Equations, parts of Solid Geometry, and *Rigid* Dynamics. When these are introduced, they must find a place in the latter part of the course.

The modern tendency seems to be to abolish the divisions between the different sections of Mathematics, and to teach them together in such order as seems to promise a logical building up of a firm structure of mathematical knowledge. Such a system 'recognises the logical unity of the material of study, and is, in fact, more scientific than the system at present in vogue.'

I have not taken up the subject of Theoretical Mechanics, as only a preliminary course is ever given in the Mathematical Department, and that only in one or two Universities. In most cases it is given in the Department of Physics.

The subject is so closely associated with Applied Mechanics, that no one could properly teach the latter without having made a special study of the theory, therefore this must be an exception to the recommendation as to the selection of a teacher.

In my opinion, the theoretical and applied mechanics, the preliminary work as well as the advanced, should certainly be under the charge of one and the same man, and that man should be an engineer.

It would be very well to add to the discussion of these mathematical subjects some thorough training in the making of computations, and in estimating the value of indefinite data.

In the numerical part of these computations, the careful use of the slide rule should not only be permitted, but encouraged. It is invaluable as an economiser of time, in making calculations which are only intended as a means of illustrating methods of working, and as a check to the actual calculations which should never be omitted where the matter is one of practical importance.

Here also would be the opportunity to teach the relation between the kind of problem under consideration and the degree to which accuracy should be carried in the necessary calculations.

What I mean is well exemplified by an instance that occurred in my own experience. A friend of mine, wishing to ascertain the measurement of distances between the holes of his golf course, employed for the purpose one of my students--a really brilliant young mathematician.

After waiting several weeks for results, the student was asked how he was getting on, and replied that he could now guarantee the distances measured to be within 1/1000th of an inch, but another week would enable him to give the measurement within 1/2000th!

In America, as you doubtless know, there has been for some years a flourishing Society for the Promotion of Engineering Education and here the Institution of Civil Engineers has lately prepared an admirable Report on the same subject. To the Discussions of the American Society I have been much indebted, and if I might venture to suggest, I should think that it might be a plan productive of good results if your example to-day were followed by the establishment of regular discussions on the educative side of subjects taken up by this and the various Engineering Societies.

THE DISCUSSION.

The Chairman (Dr. Macaulay) in proposing a vote of thanks to Prof. Bovey, said he was very grateful indeed for his very interesting address. He thought all would like to know what were the best means of preparation for boys in schools who intended to become engineers. He had no doubt that a great many ideas had been gathered on that point, from what Dr. Bovey had said. The paper covered so wide a range that it was impossible to refer to it in detail. Dr. Bovey's question as to whether text-books were all-sufficient was of special interest to him. He thought the question would not apply to boys in schools in the way that it would concern those who were old enough to proceed to technical institutions or to a university. Although he believed many would not agree with him, he was of opinion that a text-book was very nearly all-sufficient if it were only good enough. He was sorry to say that he spent a great amount of time every year in trying to find better text-books, but in most subjects he had failed to find text-books that satisfied him. His method of working was to make the boy read his text-book, while he (Dr. Macaulay) did little more than point out mistakes. He thought the best results could be obtained from telling the individual boy where he was wrong. One knew what the special needs of an individual pupil were, and, in an indirect way, it was easy to impart knowledge beyond the range of the text-book. His own experience was that he could get

through the work of an advanced Form as quickly by taking the pupils individually as by grouping them into twos and threes, because it was not necessary to say as much to the individual as to a group.

With regard to the reference to the training in speed and accuracy, he would like to say that he believed far too great a premium was placed upon speed in this country. In reforming the Mathematical Tripos, the University of Cambridge had done a great work towards remedying that state of affairs. But although at Cambridge the contest is now for class, and not for place, the premium on speed still applied to candidates for a mathematical scholarship. In spite of efforts to the contrary, it would be found that in this connection papers were becoming longer rather than shorter. At the University the slow worker would in future have as much chance as the quick worker of getting a First-Class in the Mathematical Tripos. He was completely unable to see why this should not be so, because, in the practical application of studies in after life, plenty of time could be found for working out solutions; and therefore speed was not all-important. He had no experience of what the engineer actually required, but he thought that the question of accuracy was of far greater importance than the time taken in working the result.

Mr. C. Godfrey said there was one point in regard to the preparation of a boy for an engineering career about which he had always felt somewhat dubious. He had always wondered whether mathematical training encouraged a man to visualize his work, to have a feeling for the actual machine with which he was dealing. Might he illustrate his meaning by a reference to the gyroscope? He was quite sure that he personally had no feeling for the motion of a gyroscope, and that he could not predict what the instrument was going to do. He observed that it performed a certain gyration, and, by considering the equations in regard to energy and so forth, he might, with great labour and expenditure of time, account for the peculiarity; but he was still doubtful whether that capacity was really what the engineer wanted. Was not his want really a trained instinct for the behaviour of the machine, apart from the argument of equations? Ought he really to be dependent on the equations? Must not he rather visualize and sympathize with the motions? He asked the question whether Mathematics does not rather inhibit, does not fail to develop, or even perhaps prevent the development of that type of mind. He would like to hear the opinion of an engineer on this point.

Mr. A. W. Siddons said that with regard to the question of text-books, he remembered a story being told him by a certain mathematical master who was asked why he did not use a

particular text-book on Elementary Mathematics. He was told that a boy (whose age he did not recollect) had sat down and worked at this book for six months without asking a single question. And this was held up as an advertisement for the book! The master replied that he certainly should not adopt the book, because if it had not stimulated the boy to ask a single question, it had certainly given him no ideas. He thought that boys never learned very much in that way. His own opinion was that text-books were far from being all-sufficient.

Before it was known whether Dr. Bovey would accept the invitation of the Council to read the paper they had just heard, he (Mr. Siddons) suggested that a discussion on the use of the slide-rule would be valuable. Dr. Bovey had referred to that matter, and he thought that it would not be out of place if views were exchanged regarding experience in the use of the slide-rule in schools, especially in relation to the age at which it might wisely be introduced to the boy. At Harrow they allowed it to be used quite freely by the boys on some sides of the school. In the Army Class, for instance, it was used for practically the whole time; but on the Classical and Modern sides only a few boys used it. He thought the chief objection to the early adoption of the slide-rule was that it is an expensive instrument which is easily lost. The cheaper ones soon get damaged and have their graduations rubbed off. He wished masters at other schools would say how soon their boys used the slide-rule.

Lieut.-Colonel A. Cunningham, R.E., said he could recount some practical experience which he thought would illustrate Dr. Bovey's reference to the combination of text-books in an encyclopedic form. The old, long-abolished Indian Military Seminary at Addiscombe, which passed so many generations of engineers into the Indian Army, had a comprehensive text-book * of this character. It was like a selection from text-books pieced together by an eminent mathematician (the Rev. J. Cape) of days gone by. That book was of very great service, as the parts all hung together. He had found that the separate text-books used in colleges and universities were far too diffusive for the engineer, though of course the subject could be more thoroughly learned (from the more extended books) if the student had sufficient time; but to the engineer, who had so little time, the combination text-book to which he had referred was most valuable. It had one consecutive idea passing through it, and it saved the students at that academy a great deal of time. Their ages were from sixteen to twenty. Many generations of engineers had passed away; and now, in after life, having an opportunity for contrast, he thought that the subject was better

* *A Course of Mathematics*, by the Rev. J. Cape. Longmans and Co. 1844.

learned by the engineer from the single but sufficiently comprehensive book.

Mr. E. H. Butt said he should like to give his experience in regard to the matter of text-books. This experience of teaching Mathematics applied mainly to boys, and therefore he would not speak upon University teaching. Some masters confined themselves to text-books of questions only. He had found it best to give a class of boys a thorough good outline of the subject they had to study, and then to suggest difficulties that may appear in the text-books. He thought that examinations still had too great an influence over teaching, in that masters were forced to teach so as to produce certain results. He would refer to the paper set at the Matriculation Schools Examination in Middlesex. The *first* question in the paper asked the candidate to find the compound interest on £650 at 5 per cent. for 17 years. It had to be answered without the aid of logarithms. Notwithstanding this, a severe criticism and comment on the fact that so few boys had arrived at the correct result followed the examination. (Laughter.) Anybody sitting down to do a sum of that kind would get one result one day and another the next. The compensating blunders of contracted multiplication would throw him out. As some members had said, no man in his senses would put such a question. But this example was only given in passing.

With regard to the question of feeling in sympathy with the instrument one is using, he had had exactly the same experience in connection with the peculiar action of the gyroscope. He was delighted when an old Harrow boy came to his house and showed him that he could work out and prove the Mathematical explanation of its movements. That young fellow was purely and simply a mathematician. He went to India and became a judge; but although a great part of his time was spent in judicial work he had done an enormous amount of practical engineering, the irrigation of thousands of square miles, bridge-building, and work of that description. That was the outcome of Mathematics. He (Mr. Butt) thought that was great testimony. He was of opinion that their chief object should be to bridge over the difficulty between Practical and Pure Mathematics. There was room for a good text-book in which the essential points should be picked out and all extraneous matter omitted. A number of boys he had had to do with had distinguished themselves in Practical Mathematics. One was a professor in connection with London University, another in connection with the Manchester University, some had gone through a course of Pure Mathematics (generally at Cambridge), and some had afterwards gone to Germany for two or three years' practical work. That was, he thought, a very satisfactory indication of the relative value of methods.

The Chairman said he thought that Lieut.-Colonel Cunningham's remarks were very interesting. He had never heard of such a book as that referred to by him. It seemed that there was room for a work of the kind at the present day. It would be of great use, not only for boys who were preparing for the Army, but also for those who were working for various scientific careers, because they would acquire a greater range of Mathematics in less time. He thought that if the regulations of the Army Council resulted in the production of such a book, it would be the one good thing they had done for education.

Dr. Bovey replied that he was unfortunately one of those who had perpetrated text-books. He thought that it would be a great advantage, as he had ventured to suggest, if much of the matter, now included in text-books, could be transferred to an encyclopædia of mathematics, but until such a publication was undertaken he did not see how the text-books could be much simplified, as they had to meet the wants of both under-graduate and post-graduate students. He left Cambridge as a graduate in the mathematical tripos, and he had found throughout the whole of his professional career that any success he might have had was largely due to the mathematical foundation laid in Cambridge. He might say, too, that it was very difficult for a young man to get a position of any importance as an engineer in America, or to think he had much chance of promotion (unless he were a genius) if he had not been through a well-organized technical school. In such schools in Canada and the United States, Mathematics was really the foundation of the whole work, and in some of the Universities, in which engineering is a special feature, it would be found that the students were expected to carry their mathematical training to a very high standard indeed, such special branches as Rigid Dynamics, Theory of Equations, Differential Equations, etc., being included. Another important point to remember is that a young man cannot expect to succeed unless, in addition to his technical training, he has common-sense. Common-sense was almost as important as training, and it was by the possession of this valuable characteristic, he thought, that the pure mathematician referred to by Mr. Butt proved so able a judge and so good a bridge-builder.

He hoped that he was not misunderstood with regard to the question of the importance of text-books. He did not wish it to be thought that he considered text-books the prime factor in training. Another evil he would like to refer to was the number of examinations—a necessary evil, possibly. It would be an advantage if they could be reduced to a minimum, so that students might not feel the pressure, as they usually do in the present day, of fresh examinations always ahead of them.

As regarded the question of speed, he did not at all agree with the idea that speed is of no importance. To the engineer, training in speed was of the utmost importance. He is often called upon to decide with great rapidity a problem presented on the spot, and unless his brain has been trained to work accurately and quickly he would be found wanting. The man who would succeed must always be ready to seize his opportunity. He might illustrate this by the case of a young man, still in his twenties, who on graduating was appointed to a post in one of the great Electric Companies at Brooklyn. He had been there about six months when one day the general manager of the whole system told him that the Superintendent of the New York and the Brooklyn stations had gone away at a moment's notice and would not return, and asked him if he could carry on the work for a month. Well, there was his chance. Should he take advantage of it? He was very young, but he replied that he would do his best. Within a few months he was placed at the head of twelve stations, at a salary which he (Dr. Bovey) thought every teacher would like to get.

He had always found his training in mathematics of the greatest value in dealing with hydraulic problems, and while he was in Canada he did a good deal of practical engineering in the way of water development. He had found little difficulty in modifying hydraulic equations and principles to meet special circumstances. Sometimes, perhaps, it had been necessary to stretch these modifications to an extreme limit, but still these equations would always give the power of dealing with the problems in a more scientific manner than any other method.

He would certainly like again to emphasize the importance of the use of the slide-rule, if its adoption were not adding to the numerous frills which seemed to be constantly added to different school courses. If a little time for the use of the slide-rule could be spared, the student would find it of great advantage when he entered a technical institution, because a knowledge of its use would enable him to deal with problems within a degree of accuracy demanded by practical conditions. A slide-rule could always be carried in the pocket, and the sooner a young fellow understood it, the better for him. In many Universities, the use of the slide-rule right throughout the course was insisted upon, and the plan was found to be successful. The student worked quickly and accurately.

He thought that if there were a mutual interchange of opinions between different countries as to the different systems in vogue, it might be possible to come to some harmonious decision which would make a common technical training in mathematics a possibility. (Applause.)

REVIEWS.

Practical Integration. By A. S. PERCIVAL, M.A. Pp. 86. Price 2s. 6d. net. (Macmillan & Co., 1907.)

In this book, which is attractively got up, the principal methods of integration are clearly set forth, and their utility illustrated by examples on areas, volumes, centres of gravity, moments of inertia, etc.

The view of Integration as a method of effecting a summation is developed very slightly: and no allusion is made to any of the difficulties which are involved in exhibiting the limit of a sum as an integral. On page 36 it is stated that if one is reduced to expanding the integrand in a series, and integrating term by term, "it is absolutely necessary that the series should be a convergent one." Did not the author, when he wrote thus, forget for one moment the blessed word "practical" on his title page?

By repeated integration by parts we find

$$\int_x^\infty e^{-x^2}dx = e^{-x^2}\left\{\frac{1}{2x} - \frac{1}{4x^3} + \frac{1.3}{2^3x^5} - \frac{1.3.5}{2^4x^7}\cdots\right\},$$

and the series on the right is divergent.

But the error in taking the sum of any number of terms as the value of the integral is less than the first term omitted, and so the method is of some practical utility, although only the condition stated can ease the conscience of the rigorist. C. S. J.

Introductory Mechanics. By E. J. BEDFORD. Pp. x, 141. Price 1s. 6d. (Longmans, 1908.)

This is in many respects an excellent little compilation. The diagrams are clear, the technical examples of principles are well chosen; and if sometimes we find a rule where we should like to find a reason, the limitations of 140 pages must be borne in mind.

But in one respect the book is unfortunately typical of a class. The 'practical man' has a severe training under the hardest of masters in accuracy of work. No one knows better that a rod $1\frac{1}{8}$ at one end and $1\frac{1}{8}$ at the other is not a cylinder $\frac{1}{8}$ inch in diameter. But for some reason he does not appreciate the importance of accuracy of language.

The present work is above the average of 'practical' books in carefulness of style, but even here we find on p. 14, after a definition of force based on Newton's first law, a list of various forces, viz. heat, electricity, gravity, and muscular force. Next follow directions for representing a force by a straight line. C. S. J.

Graphics: applied to Arithmetic, Mensuration and Statics. By G. C. TURNER, B.Sc. Pp. ix, 388. 1908. (Macmillan & Co.)

At the very beginning of Mr. Turner's book is a statement to which exception must be taken.

"The standard scale . . . should be divided into fifthieths of an inch and . . . millimetres or half millimetres."

No young student should ever possess or use a scale with any finer division than tenths of an inch or perhaps millimetres.

The use of an excessively fine scale (*a*) is very injurious to the sight; (*b*) continually causes mistakes in counting; (*c*) prevents the student from acquiring the valuable habit of judging subdivisions accurately by eye.

The excellent diagrams form a striking feature of the book, which consists in great part of worked-out problems. In fact, the criticism one is driven to make is that the book is overweighted by examples. The principles of graphical methods are few in number, and once they are clearly stated the student ought to make the simple and direct applications for himself.

Without going so far as to advocate expounding the principles of graphic statics on the traditional half sheet of notepaper, one may suggest that to devote 370 closely printed pages to the treatment of examples by purely graphical, excluding analytical methods, is to err by excess. C. S. J.

Festschrift zur Feier des 200 Geburtstages Leonhard Eulers. (Leipzig and Berlin: Teubner, 1907.)

It is probably safe to say that few are familiar with more than the salient features of the original work of the great eighteenth-century mathematicians. In the case of Euler, whose writings, according to Hagen's list, number upwards of seven hundred, and deal with all branches of mathematics then known, the labour required to obtain information at first hand is prohibitive; and a modern mathematical curriculum teaches us to associate Euler's name merely with a formula for the curvature of a surface, with a certain constant, perhaps with a set of hydrodynamical equations. The twenty-fifth volume of the series, *Abhandlungen zur Geschichte der Mathematischen Wissenschaften*, will therefore be widely welcomed by mathematicians, since its primary object is to supply a complete, though brief, account of Euler's work in pure mathematics.

The labour involved in the production of a volume of this kind must evidently be very great, if only on account of the compression necessary to prevent unwieldiness. It is here especially apparent in Professor Müller's article on Euler's original work. The author treats his subject under twelve heads, such as Algebra, Series, Analytical and Differential Geometry. Many of these are evidently used for convenience, for it is impossible to separate formally, say, Theory of Functions from Series; but most readers will consider that even this plea will hardly justify Analysis Situs being grouped with Combinations and Probability. Leaving this aside, however, the author is to be congratulated on the admirable way in which he has moulded all information of importance into the limited space at his command. Owing to his minute care in quoting original memoirs, he has produced a work which will, in many cases, be of greater value than a catalogue for purposes of reference.

The articles of Professors Kneuer and Lampe are of narrower scope, being designed to give an account of particular branches of Euler's work. The former deals with his contribution to the calculus of variations, and appends three small articles treating Eulerian problems from a modern point of view. These articles, coming as they do from the pen of one who has done so much for the advancement of the subject, will be of more than historical interest to all students of the calculus. The section in charge of Professor Lampe will appeal to a very wide circle of mathematical readers. It is designed to show the genesis of the idea of the logarithm of a complex number, by printing extracts of Euler's correspondence on the subject, firstly with John Bernoulli, and later with D'Alembert. So far as I am aware, this correspondence has not been easily accessible hitherto. It may therefore be hoped that these well chosen extracts will serve to establish a wider recognition of Euler's pioneer work on the modern theory of functions. It is true that, measured by the standard of Cauchy or Weierstrass, he made little progress; but it must not be forgotten that it was he who bridged the chasm of obscurity and paradox which separated the notion of the real logarithm of a real number from that of the many valued logarithm of a complex number in the early decades of the eighteenth century. J. MERCER.

Leçons sur les théories générales de l'Analyse. By RENÉ BAIRE. Gauthier-Villars. 1907 (t. 1). 1908 (t. 2).

This treatise is based on courses of lectures on Analysis delivered at the University of Dijon; as the title implies, the author's attention is directed chiefly towards *general* methods, leaving special applications on one side.

Starting with a chapter on irrational numbers (defined after Dedekind's method), limits and continuity, the book gives a good groundwork for the Calculus, Differential Equations and Analytical Functions; it concludes with a chapter on Elliptic Functions.

The change of variables in double and triple integrals is fully discussed, and the work in this connexion is very interesting, although perhaps not specially easy; and it is hardly necessary to say that M. Baire devotes special care to the arithmetical definitions of an area and of the length of a curve, which are discussed in the first volume (ch. 3). In the second volume (ch. 4) the theory of uniform convergence has been simplified by the systematic use of normal convergence; M. Baire proposes to call a series of variable terms $\Sigma f_n(x)$ *normally*

convergent in an interval (a, b), when there is a convergent series of *constants* ΣM_n, such that

$$|f_n(x)| < M_n, \quad (a \leqq x \leqq b).$$

It has, of course, been known for a long time that a normally convergent series is also uniformly convergent (Weierstrass's test*); and also that in some respects the theory of normally convergent series is easier to teach than the theory of uniformly convergent series. But it has been left to M. Baire to make the remark that *the terms of every uniformly convergent series can be grouped in brackets so that the series of brackets becomes normally convergent*; the proof is almost an immediate corollary from the definition of uniform convergence (see p. 30, t. 2 of M. Baire's *Leçons*).

The reader may now be tempted to ask—Why should we retain the use of uniform convergence, instead of replacing it by the more easily grasped idea of normal convergence?

To this question the reply would be that, unless a series is already normally convergent, it is necessary to prove that the series converges uniformly before we can proceed to arrange the terms in brackets so as to give a normally convergent series. For example the series

$$\sin x + \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x + \ldots$$

converges uniformly in the interval $(a, 2\pi - a)$ if a is any angle between 0 and $\frac{1}{2}\pi$, a result which is at once evident by using Dirichlet's test;* but it is not easy to give *directly* a simple rule for grouping the terms so that the new series converges normally.

It is perhaps worth while also to remark that a statement corresponding to M. Baire's is *not* true for uniformly convergent infinite integrals; so that in the theory of integrals uniform convergence remains indispensable.

M. Baire's course will be found well worth reading, even by those who are already acquainted with the French works on Analysis which have appeared in the last fifteen years. T. J. I'a BROMWICH.

Vorlesungen über die Weierstrass'sche Theorie der irrationalen Zahlen. By Dr. VICTOR VON DANTSCHER. Teubner. 1908. (79 pp.)

As is now generally known, it is impossible to place the theory of convergence on a firm footing without a preliminary discussion of the irrational number. Satisfactory arithmetical theories of irrationals have been developed by Cantor, Dedekind, Méray and Weierstrass; and the methods followed by the first three of these writers have been accessible for some time, but an authoritative edition of the lectures given by Weierstrass is still out of our reach.

However, Dr. v. Dantscher's lectures contained in the book under notice are based on two courses delivered by Weierstrass (in 1872 and 1884), and careful indications are given as to the work which is taken directly from Weierstrass. The fundamental notion in the present theory is the *convergent additive aggregate*, consisting of an infinity of positive rational numbers, any finite number of which are to be regarded as combined by addition; to ensure convergence, it must be possible to find a rational number K which is greater than the sum of any number of terms of the aggregate. The convergent aggregate is regarded as defining a number: it is then necessary to build up the laws of algebra for such numbers, and to prove that these laws are the same as for rational numbers.

We must heartily welcome this account of the methods adopted by Weierstrass, even though some of us may prefer to follow Dedekind's or Cantor's procedure in actual teaching. T. J. I'a BROMWICH.

QUERIES.

(61) Is there any book of Progressive Mathematical Exercises for home work, similar to that of Mr. A. T. Richardson, but on more modern lines?
H. F. D. TURNER.

(62) A reference library is being started at a school, and £2 has been voted for Mathematical books. The work may reach Scholarship standard. What is the best selection? H. F. D. TURNER.

* For these tests of uniform convergence see pp. 113, 114 of my *Infinite Series*.

ANSWERS TO QUERIES.

[10, p. 95.] It does not seem to have been noticed that the true envelope of these parabolas is the line at infinity, and *is* present in the discriminant; this fact is evident at once if we use homogeneous coordinates.

Then the parabolas are of the form

$$\lambda^2 yz+\mu^2 zx+(\lambda+\mu)^2 xy=0,$$

using areal coordinates; and the discriminant is

$$xyz(x+y+z)=0.$$

As remarked by Mr. Richmond, if we write $x=0$ in the equation to the set of parabolas, we get the equation $\lambda^2=0$, thus justifying the presence of the factor x; and similarly for y and z.

Other illustrations of the point involved in this example will be found in Chrystal's paper on the Singular Solutions of Differential Equations (*Trans. Roy. Soc. Edin.*, vol. 38, 1896, p. 803).

One of Chrystal's examples is the set of cubics

$$z^3+\lambda^3 xy(x+y)-3\lambda^2 xyz=0,$$

for which the discriminant is

$$x^2y^2(x-y)^2z^6=0.$$

Here x, y are inflexional tangents to all the cubics, and so may be called the envelope; $x-y=0$ is a *node-locus*,* and the presence of z is to be accounted for by the method explained in Mr. Richmond's note.

T. J. Pa. Bromwich.

[23, p. 131.] An elementary proof of the approximation

$$\log(n!)=(n+\tfrac{1}{2})\log n-n+\tfrac{1}{2}\log(2\pi)+\frac{1}{12n}-\frac{\theta}{360n^3}$$

(where $0<\theta<1$) was given in the *Messenger of Mathematics*, October 1906, p. 81. This approximation is sufficiently close for five-figure logarithms for any value of n not less than 10, since the remainder term is then less than $\frac{1}{3}10^{-5}$.

For a discussion of the asymptotic series for $\log(n!)$ or $\log\Gamma(1+x)$, I may refer to my book *Infinite Series*, Arts. 132 and 176.

T. J. Pa. Bromwich.

[26, p. 166.] The equation given by Gauss,

$$\Gamma(1+x)=\lim_{n\to\infty}\frac{n^x.\,n!}{(1+x)(2+x)\ldots(n+x)},$$

defines the Gamma-function for all values of x (complex as well as real) except for negative real values. Euler's integral also can be applied if the real part of $1+x$ is positive; and complex integrals can be found which are valid for all values of x.

[28, p. 166.] In addition to the constructions given by Mr. Lodge, it may be noted that if D, E, F are derived from ABC by drawing equilateral triangles on BC, CA, AB (all inwards or all outwards), and if P, Q, R are derived in the *same* way from DEF, then A is found by bisecting PD, etc.

On the other hand, if D is an inward vertex, E, F being outwards, the points A, D, E, F form a parallelogram of which AD, EF are the diagonals. Thus A is easily found, and then the triangle ABC is easily constructed.

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*That a node-locus appears as a squared factor in the discriminant has been proved by Hill (*Proc. Lond. Math. Soc.*, vol. 19, 1889, p. 566, § 8); see also Bromwich and Hudson (*Quarterly Journal*, vol. 34, 1902, p. 114), where it is proved further that an *osculating* envelope is a *squared* factor in the discriminant (p. 111).

[32, p. 166.] For the sum of the squares of the coefficients of $(1+x)^n$, when n is not necessarily a positive integer, we have the formula

$$\Gamma(2n+1)/\{\Gamma(n+1)\}^2.$$

If the signs of the alternate squares are made negative, we find the sum

$$\cos(\tfrac{1}{2}n\pi)\Gamma(n+1)/\{\Gamma(\tfrac{1}{2}n+1)\}^2.$$

An investigation of these will be found in my *Infinite Series*, p. 249, Ex. 21. The sum of the cubes of the coefficients in $(1-x)^{-p}$ is given by the formula

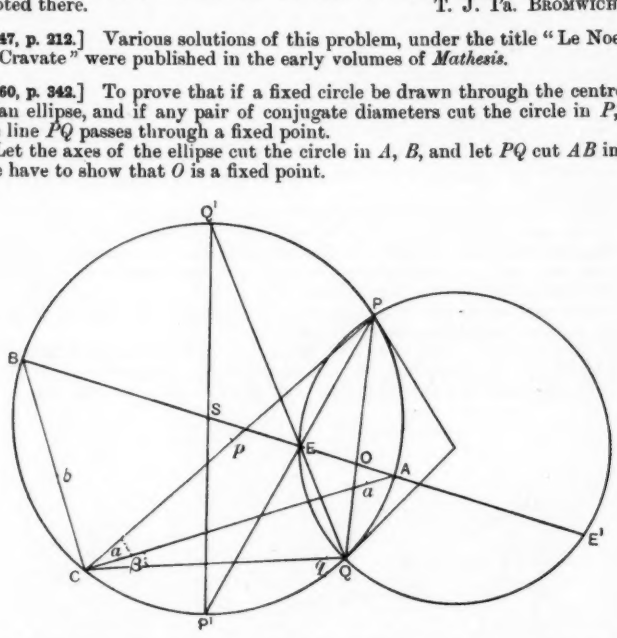
$$\cos(\tfrac{1}{2}p\pi)\Gamma(1-\tfrac{3}{2}p)/\{\Gamma(1-\tfrac{1}{2}p)\}^3,$$

proved by Morley (*Proc. Lond. Math. Soc.*, vol. 34, p. 397) and other writers quoted there. T. J. I'a. BROMWICH.

[47, p. 212.] Various solutions of this problem, under the title "Le Noeud de Cravate" were published in the early volumes of *Mathesis*.

[60, p. 342.] To prove that if a fixed circle be drawn through the centre C of an ellipse, and if any pair of conjugate diameters cut the circle in P, Q, the line PQ passes through a fixed point.

Let the axes of the ellipse cut the circle in A, B, and let PQ cut AB in O. We have to show that O is a fixed point.



Let $\angle PCA=\alpha$, and $ACQ=\beta$, then $\tan\alpha\tan\beta=b^2:a^2$, where a, b are the semi-axes of the ellipse.

Now
$$\frac{OA}{QO}=\frac{\sin OQA}{\sin QAO}=\frac{\sin\alpha}{\cos\beta},$$

and
$$\frac{QO}{BO}=\frac{\sin QBO}{\sin BQO}=\frac{\sin\beta}{\cos\alpha},$$

$$\therefore\ \frac{OA}{BO}=\tan\alpha\tan\beta=\frac{b^2}{a^2},$$

i.e. O is a fixed point dividing BA in the ratio $a^2:b^2$.

The above neat proof was given me last September by Mr. Oskar Becker, now at New College, Oxford. The circle is well known in connection with the ellipse, particularly when its diameter is $a\pm b$. In that case a point can

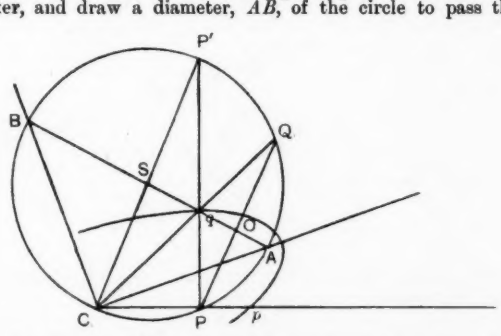
be found on it which will trace the ellipse when AB is made to slide on the fixed lines CA, CB. To find this tracing point, describe a circle cutting the given circle orthogonally in P, Q, and cutting AB internally in E and externally in E'. Then if $AB=a+b$, E is the tracing point; and if $AB=a-b$, E' is the tracing point. For if two circles cut orthogonally, any diameter of one is cut internally and externally by the other in a ratio which is the sub-duplicate ratio of the segments of the diameter made by the common chord: $\therefore\ BE : EA = a : b = -BE' : E'A$.

When E is found (taking $AB=a+b$ for definiteness), the length of any semi-diameter of the ellipse can be easily found without actually sliding AB. Thus if any semi-diameter Cp cuts the given circle in P, join PE and produce it to cut the circle again in P', then $EP'=Cp$. For if AEB slides on the fixed lines CA, CB, carrying the circle and all its chords with it, PEP' will slide on the fixed lines CP, CP' (owing to the constancy of the angles at C subtended by the circular arcs AP, AP', etc.); therefore when P' slides down to C, PEP will lie along CP, and E must coincide with p. The other part, EP, of the chord PP' is also of interest, for, if the ellipse is the ellipse of inertia of a plane lamina, EP, being inversely proportional to EP', is the radius of gyration about the line CP. [See *Phil. Mag.* November, 1886: "Geometrical representation of Moments and Products of inertia in a plane section," etc.]

If Cp, Cq are conjugate diameters, the line $P'Q'$ is a diameter of the circle, for it is easily seen that $PEQ=90°+PP'Q$; $\therefore\ PQQ'=90°$. The lengths of the semi-axes Ca, Cb are EB, EA respectively. By drawing an ellipse in this way with pairs of conjugate diameters, we obtain not only points on the curve, but also the tangents at those points.

I have for many years used this circle to construct the axes of an ellipse when a pair of conjugate diameters are given:

Thus, if Cp, Cq are conjugate semi-diameters, draw qP perpendicular to Cp, and produce it backwards to P', so that $qP'=Cp$. Describe a circle on CP' as diameter, and draw a diameter, AB, of the circle to pass through q.



Then CA, CB are the directions of the axes, and qB, qA are their semi-lengths. The construction is merely a particular case of the general one given above, q being the tracing-point E. The method is a modification of one given in Captain Eagles' "Constructive Geometry of Plane Curves," which is a storehouse of excellent constructions. If Cq is produced to meet the circle again in Q, PQ will meet AB in the fixed point O, which is in this case found without the aid of an orthogonal circle.

It is interesting to note that if a line AB is divided in any ratio $AK : KB$, the construction enables us readily to divide the line in the duplicate ratio. Thus, describe a circle on AB as diameter, and join K to the ends of any

other diameter, producing the joining lines to meet the circle in P, Q; then PQ cuts AB in the ratio $AK^2 : KB^2$. And the former construction, by help of the orthogonal circle, would enable us to divide AB in the ratio $\sqrt{AK} : \sqrt{KB}$. There are, of course, other methods, but these are very simple.

This solution answers (1) of Query 60, and gives some answer to (3), but leaves (2) still unanswered, substituting a method of finding the actual length of the diameter instead. A. LODGE.

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